Analysis on flat symmetric spaces

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ANALYSIS ON FLAT SYMMETRIC SPACES

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Abstract. By taking an appropriate zero-curvature limit, we obtain the spherical functions on flat symmetric spaces G_0/K as limits of Harish-Chandra's spherical functions. New and explicit formulas for the spherical functions on G_0/K are given. For symmetric spaces with root system of type A_n, we find the Taylor expansion of the spherical functions on G_0/K in a series of Jack polynomials.

Résumé. Dans cet article nous obtenons les fonctions sphériques des espaces symétriques plats G_0/K comme une limite des fonctions sphériques de Harish-Chandra. Des formules nouvelles et explicites pour les fonctions sphériques sur G_0/K sont données. Dans le cas des espaces symétriques de type A_n, nous parvenons à donner le développement en série de Taylor des fonctions sphériques sur G_0/K, en termes de polynômes de Jack.

1. Introduction

In this paper we shall initiate an investigation of the so-called HIZ-type integral (Harish Chandra-Itzykson-Zuber); these are Fourier transforms of orbits of K in the tangent space at the origin of a semi-simple non-compact symmetric space G/K, where G is a connected non-compact semi-simple Lie group with finite center, and K is a maximal compact subgroup. These integrals play an important role in the theory of integrable systems in physics, and in connection with the study of random matrices. It is well-known that they correspond to spherical functions on the tangent space, viewed as a flat symmetric space. Our point of view is to see the HIZ-type integrals as limits of spherical functions for G/K, and we are able to obtain new and explicit formulas by analyzing the deformation (as the curvature goes to zero) of G/K to its tangent space. At the same time we develop the connection to the Calogero-Moser system by showing that this is in a simple way conjugate to the limit of the radial part of the Laplace-Beltrami operator on G/K. We plan to develop this point of view further in connection with generalized Bessel functions in a sequel to this paper [5].

To be more specific about our results, let $\mathfrak{g}$ be the Lie algebra of G. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, and a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. It is well known that the spherical functions $\varphi_\lambda(g)$ on G/K are eigenfunctions for the radial part of the Laplace-Beltrami operator on G/K, with eigenvalue $-\big(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\big)$. After a contraction process on $\mathfrak{g}$, we study the limit behavior of $\varphi_\lambda(g)$ as we let the curvature of G/K tend to zero. In particular, we prove that the following limit

$$\psi(\lambda, X) := \lim_{\varepsilon\to 0} \varphi_{\frac{\lambda}{\varepsilon}}\big(\exp(\varepsilon X)\big), \qquad X \in \mathfrak{p},$$

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exists, and its integral representation (in the standard notation) is given by

$$\psi(\lambda, X) = \int_K e^{iB(A_\lambda, \mathrm{Ad}(k)X)} dk.$$

Moreover, we prove that $\psi(\lambda, X)$ exhausts the class of spherical functions on the flat symmetric space $G_0/K \simeq \mathfrak{p}$, where $G_0 = K \ltimes \mathfrak{p}$. In particular, $\psi(\lambda, X)$ is an eigenfunction for the radial part Δ_0, of the Laplace-Beltrami operator on G_0/K, with eigenvalue $-\langle \lambda, \lambda \rangle$. The contraction principle was used earlier in [13], for understanding the relationship between the representation theories of $K \ltimes \mathfrak{p}$ and G. The limit approach was also used in [37] to define the Weyl transform on flat symmetric spaces where G/K is a Hermitian symmetric space.

The spherical functions $\psi(\lambda, X)$ are also called generalized Bessel functions. The literature is rich by its applications to analysis on symmetric spaces. For instance, in [23] and [24], Helgason proved Paley-Wiener type theorems for the spherical transform and for the generalized Bessel transform on G_0/K. In [38] Opdam studied these spherical functions from another point of view.

It is remarkable that in spite of many results about the analysis of spherical functions on G/K, their Fourier analysis and asymptotic properties, it is only for very few cases that explicit formulas exist for these functions. For flat symmetric spaces we shall see that one may derive at least the same amount of explicit information by a limit analysis, when the curvature goes to zero, as for spherical functions; and in some cases even more information. For instance, when G/K admits a root system of type A_n, we obtain the Taylor expansion of $\psi(\lambda, X)$ in a series of Jack polynomials.

In the case when G is complex, $\psi(\lambda, X)$ is the so-called Harish-Chandra integral. Note that our approach of defining $\psi(\lambda, X)$ gives an alternative and simple proof for the Harish-Chandra integral.

Explicit formulas, or Taylor expansions, of $\psi(\lambda, X)$ have significant applications in random matrices theories. We refer to [3, 4] for discussion on this issue. Explicit formulas are also very useful in the study of the Radon transform on flat symmetric spaces [44].

In connection with the radial part Δ_0 of the Laplace-Beltrami operator on the flat symmetric space G_0/K, we discus a class of integrable model named Calogero-Moser model. We prove that the following Hamiltonian

$$\mathscr{H} = L_A + \frac{1}{4} \sum_{\alpha > 0} m_\alpha (2 - m_\alpha - 2m_{2\alpha}) \frac{\langle \alpha, \alpha \rangle}{\alpha^2}$$

(in the standard notation) is conjugate to the radial part Δ_0. As a natural consequence we obtain the eigenfunction for $\mathscr{H}$ with eigenvalue $-\langle \lambda, \lambda \rangle$. (After the paper was completed, Rösler pointed out for us the reference [41] were she has also proved the same statement.)

This paper is organized as follows: In section two we introduce some notation and preliminary results on spherical functions for symmetric spaces of the noncompact type. In the third section we establish the deformation principle and use it to identify the HIZ-type integrals with limits of spherical functions on G/K; and in section four we show the connection to the Calogero-Moser system. In section five we give explicit HIZ-type integral generalizing previous formulas in the physics literature. In the sixth section we use results of Hoogenboom to get the HIZ-type integral for the case of $G = SU(p, q)$. In the last sections, we treat in detail the rank one case, the rank two case, the complex case, and we develop a Taylor formula for the HIZ-type integrals for the case of symmetric spaces with root system of type A_n

2. Notation and preliminary results

Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. The symmetric space G/K is a Riemannian symmetric space of non-compact type.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{k}$ being the Lie algebra of K, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $B(\cdot,\cdot)$ of $\mathfrak{g}$. Set θ to be the corresponding Cartan involution.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$, and W be the Weyl group acting on $\mathfrak{a}$. This action extends to the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, and to the complexification $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Fix a Weyl positive chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let Σ^+ be the corresponding set of positive roots. We denote by ρ the half-sum of positive roots $\alpha \in \Sigma^+$ with multiplicities m_α counted. Let $\mathfrak{n}$ be the sum of the root spaces $\mathfrak{g}^{(\alpha)}$ corresponding to the positive roots. The connected subgroups of G associated with the subalgebras $\mathfrak{a}$ and $\mathfrak{n}$ are denoted by the corresponding capital letters. We have the Iwasawa decomposition $G = KAN$. For $g \in G$, define $H(g) \in \mathfrak{a}$ by $g \in K\exp(H(g))N$.

Let $\mathscr{D}(G/K)$ be the algebra of G-invariant differential operators on G/K.

Suppose the smooth complex-valued function φ_λ is an eigenfunction of each $D \in \mathscr{D}(G/K)$

$$D\varphi_\lambda = \gamma_D(\lambda)\varphi_\lambda, \qquad \lambda \in \mathfrak{a}^*.$$

Here the eigenfunction is labeled by the parameters λ, and $\gamma_D(\lambda)$ is the eigenvalue. If in addition φ_λ satisfies $\varphi_\lambda(e) = 1$, where e is the identity element, and $\varphi_\lambda(kgk') = \varphi_\lambda(g)$ for $k, k' \in K$, then φ_λ is called a spherical function. Because of the bi-invariance under K, these functions are completely determined on the radial part A.

Let Δ be the radial part of the Laplace-Beltrami operator on G/K. Then

$$\Delta = \mathscr{L}_A + \sum_{\alpha \in \Sigma^+} m_\alpha (\coth \alpha) A_\alpha,$$

where m_α is the multiplicity of the root α, $\mathscr{L}_A$ is the Laplacian on A, and $A_\alpha \in \mathfrak{a}$ is determined by $B(A_\alpha, H) = \alpha(H)$ $(H \in \mathfrak{a})$. In particular

$$\Delta\varphi_\lambda = -\big(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\big)\varphi_\lambda.$$

In [19], Harish-Chandra proves the following integral representation of the spherical functions.

Theorem 2.1. *As λ runs through $\mathfrak{a}_{\mathbb{C}}^*$, the functions*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda-\rho)H(gk)}\,dk, \qquad g \in G,$$

exhaust the class of spherical functions on G. Moreover, two such functions φ_λ and φ_μ are identical if and only if $\lambda = \omega\mu$ for some ω in the Weyl group W.

3. Deformation of the Lie algebra $\mathfrak{g}$

For $\varepsilon \geq 0$, set $\mathfrak{g}_\varepsilon := \mathfrak{k} \oplus \mathfrak{p}$ with Lie bracket $[\cdot,\cdot]_\varepsilon$, such that

$$\begin{aligned} &[X, X']_\varepsilon = [X, X'], && X, X' \in \mathfrak{k}; \\ &[Y, Y']_\varepsilon = \varepsilon^2 [Y, Y'], && Y, Y' \in \mathfrak{p}; \\ &[X, Y]_\varepsilon = [X, Y], && X \in \mathfrak{k},\ Y \in \mathfrak{p}. \end{aligned}$$

Here $[\cdot,\cdot]$ denotes the Lie bracket associated with $\mathfrak{g}$.

Lemma 3.1. (i) *For $\varepsilon > 0$, the Lie algebra $\mathfrak{g}_\varepsilon$ is isomorphic to $\mathfrak{g}$. If $\varepsilon = 0$, $\mathfrak{g}_0$ is a non-semisimple Lie algebra.*

(ii) *If $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, then $\varepsilon\alpha \in \Sigma(\mathfrak{g}_\varepsilon, \mathfrak{a})$.*

(iii) *The root space $\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)}$ associated with the root $\varepsilon\alpha$ is given by*

$$\mathfrak{g}_\varepsilon^{(\varepsilon\alpha)} = \Big\{ \varepsilon X_k + X_p \ \mid \ X_k + X_p \in \mathfrak{g}^{(\alpha)} \text{ where } X_k \in \mathfrak{k},\ X_p \in \mathfrak{p} \Big\}.$$

Proof. (i) For $\varepsilon > 0$, let $\Phi_\varepsilon : \mathfrak{g}_\varepsilon \to \mathfrak{g}$ be defined by

$$\Phi_\varepsilon(X) = X, \ \text{ if } X \in \mathfrak{k} \qquad \text{and} \qquad \Phi_\varepsilon(Y) = \varepsilon^{-1} Y, \ \text{ if } Y \in \mathfrak{p}.$$

For $X_1, X_2 \in \mathfrak{k}$

$$[\Phi_\varepsilon(X_1), \Phi_\varepsilon(X_2)]_\varepsilon = [X_1, X_2] = \Phi_\varepsilon([X_1, X_2]).$$

For $Y_1, Y_2 \in \mathfrak{p}$

$$[\Phi_\varepsilon(Y_1), \Phi_\varepsilon(Y_2)]_\varepsilon = [\varepsilon^{-1} Y_1, \varepsilon^{-1} Y_2]_\varepsilon = [Y_1, Y_2] = \Phi_\varepsilon([Y_1, Y_2]).$$

Finally, for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$

$$[\Phi_\varepsilon(X), \Phi_\varepsilon(Y)]_\varepsilon = [X, \varepsilon^{-1} Y]_\varepsilon = \varepsilon^{-1}[X, Y] = \Phi_\varepsilon([X, Y]).$$

Therefore the first statement holds.

(ii) Recall that each $X \in \mathfrak{g}^{(\alpha)}$ can be written as

$$X = X_k + X_p := \frac{1+\theta}{2}(X) + \frac{1-\theta}{2}(X) \in \mathfrak{k} \oplus \mathfrak{p}.$$

Also, recall that for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ and $H \in \mathfrak{a}$, we have $[H, X] = \alpha(H) X$. Applying θ, we obtain $[-H, \theta(X)] = \alpha(H)\theta(X)$. Hence

$$[H, X_p] = \alpha(H) X_k, \qquad [H, X_k] = \alpha(H) X_p. \tag{3.1}$$

Using (3.1) we can see that for $\varepsilon > 0$ and $H \in \mathfrak{a}$

$$\begin{aligned} [H, \varepsilon X_k + X_p]_\varepsilon &= [H, \varepsilon X_k]_\varepsilon + [H, X_p]_\varepsilon = [H, \varepsilon X_k] + \varepsilon^2 [H, X_p] \\ &= \varepsilon\alpha(H) X_p + \varepsilon^2 \alpha(H) X_k = \varepsilon\alpha(H)\{X_p + \varepsilon X_k\}. \end{aligned}$$

Hence $\varepsilon\alpha \in \Sigma(g_\varepsilon, \mathfrak{a})$.

(iii) The third claim follows from the proof of statement (ii). $\square$

Let G_ε be the analytic Lie group with Lie algebra $\mathfrak{g}_\varepsilon$ via the Baker-Campbell-Hausdorff formula, and set Δ_ε to be the radial part of the Laplace-Beltrami operator on G_ε / K. Therefore

$$\Delta_\varepsilon = \mathscr{L}_A^{(\varepsilon)} + \sum_{\alpha \in \Sigma^+} m_\alpha \coth(\varepsilon\alpha) A_{\varepsilon\alpha},$$

where $\mathscr{L}_A^{(\varepsilon)}$ is the Laplacian on A associated with an orthonormal basis $\{H_i^{(\varepsilon)}\}$ for $\mathfrak{a}$ in $\mathfrak{g}_\varepsilon$, and $A_{\varepsilon\alpha} \in \mathfrak{a}$ is determined by $B_\varepsilon(A_{\varepsilon\alpha}, H) = \varepsilon\alpha(H)$ for $H \in \mathfrak{a}$. Here $B_\varepsilon(\cdot,\cdot)$ is the Killing form for $\mathfrak{g}_\varepsilon$.

Lemma 3.2. *The following identities hold*

$$A_{\varepsilon\alpha} = \varepsilon^{-1}A_\alpha, \qquad H_i^{(\varepsilon)} = \varepsilon^{-1}H_i,$$

where $\{H_i\}$ *is an orthonormal basis for* $\mathfrak{a}$ *in* $\mathfrak{g}$.

Proof. Using Lemma 3.1, one can see that $B_\varepsilon(A_{\varepsilon\alpha}, H) = \varepsilon^2 B(A_{\varepsilon\alpha}, H)$. On the otherhand

$$B_\varepsilon(A_{\varepsilon\alpha}, H) = \varepsilon\alpha(H) = \varepsilon B(A_\alpha, H).$$

Therefore $A_{\varepsilon\alpha} = \varepsilon^{-1}A_\alpha$. In the same way, the second part of the lemma holds. □

From Lemma 3.2 it follows that $\Delta_\varepsilon = (\varepsilon^*)^{-1} \circ \Delta \circ \varepsilon^*$, where $\varepsilon^* f(X) = f(\varepsilon X)$. Another consequence of Lemma 3.2 is the following theorem.

Theorem 3.3. *Set*

$$\Delta_0 = \mathscr{L}_A + \sum_{\alpha\in\Sigma^+} \frac{m_\alpha}{\alpha} A_\alpha.$$

The following limit holds

$$\lim_{\varepsilon\to 0} \varepsilon^2 \Delta_\varepsilon = \Delta_0.$$

It is known that the symmetric spaces fall into three different categories: the compact-type, the noncompact-type, and the Euclidean-type (or flat symmetric space). The three cases can be distinguished by means of their curvature. In the class of compact-type, the symmetric space has sectional curvature everywhere positive. In the class of noncompact-type, the symmetric space has sectional curvature everywhere negative, and in the class of flat symmetric spaces, the sectional curvature is zero.

Actually, if $\mathfrak{g}$ is the Lie algebra of a Lie group G with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, then, if $G_0 = K \ltimes \mathfrak{p}$, the flat symmetric space G_0/K can be identified with $\mathfrak{p}$. The elements $g_0 = (k, p)$ act on G_0/K in the following way

$$g_0(p') = \mathrm{Ad}(k)p' + p, \qquad k \in K, \quad p, p' \in G_0/K.$$

An example of a zero curvature symmetric space is the flat Euclidean space in four dimensions. It is known that this space can be realized as the coset of the Euclidean Poincaré group $\tilde{P}$ with respect to $SO(4)$, $\mathfrak{p} \simeq \tilde{P}/SO(4)$. The translations of the Poincaré group play the role of $\mathfrak{p}$, and they are isomorphic to Euclidean space and have all the characteristics of a zero curvature symmetric space. The fact that the zero curvature spaces can be obtained as limit of positive curvature spaces can be exemplified as follows. We can realize the Euclidean Poincaré group as a suitable limit of the $SO(5)$ group. In this limit, the coset $SO(5)/SO(4)$, which is the four dimensional unit sphere, becomes the Euclidean four-dimensional space.

For the spherical functions on the three categories of symmetric spaces, it is well known that the spherical functions on symmetric spaces of the noncompact-type can be obtained from the spherical functions on symmetric spaces of the compact-type, and vice versa, via some analytic continuation. Next we prove that the spherical functions associated with flat symmetric spaces can also be obtained from the spherical functions on symmetric spaces of the noncompact-type, or the compact-type, by letting the curvature tend to zero from the left, or from the right, respectively.

For $\varepsilon > 0$, write $g_\varepsilon = k\exp(\varepsilon X)$, with $k \in K$ and $X \in \mathfrak{p}$. Denote by

$$\psi(\lambda, X) := \lim_{\varepsilon \to 0} \varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon).$$

The key ingredient involved in the proof of the following theorem was used earlier by Dooley and Rice [13] to investigate the relationship between the principal series of G and $K \ltimes \mathfrak{p}$. See also [9].

Theorem 3.4. *The limit $\psi(\lambda, X)$, and its derivatives exist. Its integral representation is given by*

$$\psi(\lambda, X) = \int_K e^{iB(\mathrm{Ad}(k)X, A_\lambda)} dk.$$

Moreover, $\psi(\lambda, X)$ satisfies

$$\Delta_0 \psi(\lambda, X) = -\langle \lambda, \lambda \rangle \psi(\lambda, X).$$

The limit $\psi(\lambda, X)$ is the so-called HIZ-type integral, and it is well known that it corresponds to spherical functions on the flat symmetric space $\mathfrak{p}$.

Proof. Denote by $\mathbb{P} : \mathfrak{p} \to \mathfrak{a}$ the orthogonal projection on $\mathfrak{a}$ for the scalar product associated with the Killing form. Since $g \mapsto \varphi_\lambda(g)$ is K-invariant, it is enough to prove the statement for $g_\varepsilon = \exp(\varepsilon X)$ with $X \in \mathfrak{p}$.

Note that $H(\exp(\varepsilon X)k) = H(\exp(\varepsilon k^{-1} \cdot X))$, where $k^{-1} \cdot X := \mathrm{Ad}(k)X$. Write $k^{-1} \cdot X = \mathbb{P}(k^{-1} \cdot X) + Y \in \mathfrak{a} \oplus \mathfrak{a}^\perp$, where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Since each $Y \in \mathfrak{a}^\perp$ can be written as $Y = Y_k + Y_n \in \mathfrak{k} \oplus \mathfrak{n}$ (cf. [22]), then $k^{-1} \cdot X = \mathbb{P}(k^{-1} \cdot X) + Y_k + Y_n$. Using the fact that

$$\begin{aligned} \exp(\varepsilon Y_k) \exp\big(\varepsilon \mathbb{P}(k^{-1} \cdot X)\big) \exp(\varepsilon Y_n) &= \exp\big(\varepsilon \mathbb{P}(k^{-1} \cdot X) + \varepsilon(Y_k + Y_n) + \mathscr{O}(\varepsilon^2)\big) \\ &= \exp\big(\varepsilon k^{-1} \cdot X + \mathscr{O}(\varepsilon^2)\big), \end{aligned}$$

one can deduce that both functions, $\varepsilon \mapsto H\big(\exp(\varepsilon Y_k) \exp(\varepsilon \mathbb{P}(k^{-1} \cdot X)) \exp(\varepsilon Y_n)\big)$ and $\varepsilon \mapsto H\big(\exp(\varepsilon k^{-1} \cdot X)\big)$, have the same derivative at $\varepsilon = 0$. Using the definition of $H(\cdot)$ for the first function, we can see that

$$\frac{d}{d\varepsilon} H\big(\exp(\varepsilon k^{-1} \cdot X)\big)_{|\varepsilon=0} = \mathbb{P}(k^{-1} \cdot X),$$

uniformly for $k \in K$ and X in a bounded set, and therefore

$$\lim_{\varepsilon \to 0} \frac{\lambda}{\varepsilon} H\big(\exp(\varepsilon k^{-1} \cdot X)\big) = \lambda \mathbb{P}(k^{-1} \cdot X).$$

Hence the integral representation of $\psi(\lambda, X)$ holds. The same idea can be employed to prove that the derivatives of φ_λ also converge to the derivatives of $\psi(\lambda, \cdot)$.

Next, using the fact that φ_λ it is an eigenfunction for the radial part (of the Laplace-Beltrami operator) Δ with eigenvalue $-\left(\langle\lambda,\lambda\rangle+\langle\rho,\rho\rangle\right)$, we obtain

$$\begin{aligned}
\Delta_\varepsilon \varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon) &= \Big\{\varepsilon^{-2}L_A + \sum_{\alpha\in\Sigma^+} m_\alpha \coth(\varepsilon\alpha)\varepsilon^{-1}A_\alpha\Big\}\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon)\\
&= \Big(L_A\varphi_{\frac{\lambda}{\varepsilon}}\Big)(g_\varepsilon) + \sum_{\alpha\in\Sigma^+} m_\alpha\coth(\varepsilon\alpha)\Big(A_\alpha\varphi_{\frac{\lambda}{\varepsilon}}\Big)(g_\varepsilon)\\
&= -\left(\left\langle\frac{\lambda}{\varepsilon},\frac{\lambda}{\varepsilon}\right\rangle + \langle\rho,\rho\rangle\right)\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon) - \sum_{\alpha\in\Sigma^+} m_\alpha\coth(\varepsilon\alpha)\Big(A_\alpha\varphi_{\frac{\lambda}{\varepsilon}}\Big)(g_\varepsilon)\\
&\quad + \sum_{\alpha\in\Sigma^+} m_\alpha\coth(\varepsilon\alpha)\Big(A_\alpha\varphi_{\frac{\lambda}{\varepsilon}}\Big)(g_\varepsilon)\\
&= -\left(\left\langle\frac{\lambda}{\varepsilon},\frac{\lambda}{\varepsilon}\right\rangle + \langle\rho,\rho\rangle\right)\varphi_{\frac{\lambda}{\varepsilon}}(g_\varepsilon).
\end{aligned}$$

Therefore, the last claim of the theorem holds. ☐

Remark 3.5. (i) Using the fact that for all g and h in G

$$\varphi_\lambda(gh) = \int_K e^{(i\lambda-\rho)H(gk)}e^{-(i\lambda+\rho)H(h^{-1}k)}dk,$$

we can see that the limit of

$$\varphi_{\frac{\lambda}{\varepsilon}}(g\exp(\varepsilon X)) \sim \int_K e^{i(\frac{\lambda}{\varepsilon}+\rho)H(gk)}e^{iB(A_\lambda,\mathrm{Ad}(k)X)}dk \qquad \text{as } \varepsilon\to 0,$$

does not exist.

(ii) Since $\varphi_\lambda=\varphi_\mu$ if and only if $\lambda=\omega\mu$ for $\omega\in W$, the same statement holds for $\psi(\lambda,\cdot)$.

(iii) The spherical functions $\psi(\lambda,X)$ are symmetric with respect to λ and X, while Δ_0 is not symmetric under interchange of λ and X.

Remark 3.6. (i) The same approach can be used to derive the Plancherel formula for the Bessel transform on G_0/K by means of the Plancherel formula for G/K.

(ii) In [38] Opdam also studied these spherical functions from another point of view. In connection with this approach see [28] and [40].

4. Connection to the Calogero-Moser system

In this section we develop the connection to the Calogero-Moser system on G_0/K by showing that this is in a simple way conjugate to the radial part Δ_0. For details on Calogero-Moser systems we refer to [6, 16, 39]. These models describe n particles in one dimension, identified by their coordinates $q_1,\ldots,q_n$. The Hamiltonian of such a system is given by

$$\mathscr{H} = \sum_{i=1}^n \frac{\partial^2}{\partial q_i^2} + \frac{1}{4}\sum_{\alpha>0} m_\alpha(2-m_\alpha-2m_{2\alpha})\frac{\langle\alpha,\alpha\rangle}{\langle q,\alpha\rangle^2}.$$

Here the coordinates are $q=(q_1,\ldots,q_n)$, and the particle mass is set to unity.

The goal of this section is to find a connection between the radial part Δ_0, of the Laplace-Beltrami operator on $\mathfrak{p}$, and the Calogero-Moser system. Also, we will investigate the solution of the following Schrödinger equation

$$\mathscr{H}\Phi_\lambda(q) = -\langle\lambda,\lambda\rangle\Phi_\lambda(q),$$

such that

$$\int_{\mathfrak{a}^+} |\Phi_\lambda(q)|^2 dq < \infty, \qquad \text{with } \mathfrak{a}^+ = \left\{ q \in \mathfrak{a} \mid \langle q, \alpha \rangle > 0 \ \ \forall \alpha \in \Sigma^+ \right\},$$

$$\Phi_\lambda(q) = 0 \quad \text{if } \langle q, \alpha \rangle = 0, \quad \text{and} \quad \Phi_\lambda(\omega q) = \Phi_\lambda(q) \quad \text{for } \omega \in W.$$

For $q \in \mathbb{R}^n$ and $\alpha \in \Sigma^+$, denote by r_α the reflection on the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α

$$r_\alpha(q) := q - \frac{2\langle q, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

Proposition 4.1. (cf. [10]) *Let $\omega \in W$ be a plane rotation, i.e. ω is a product of two reflections and $\omega \neq e$. Then as a rational function in q*

$$\sum_{r_\alpha r_\beta = \omega} m_\alpha m_\beta \frac{\langle \alpha, \beta \rangle}{\langle q, \alpha \rangle \langle q, \beta \rangle} = 0.$$

Corollary 4.2. *As a rational function in q*

$$\sum_{\substack{\alpha, \beta > 0 \\ \alpha \neq \beta}} m_\alpha m_\beta \frac{\langle \alpha, \beta \rangle}{\langle q, \alpha \rangle \langle q, \beta \rangle} = 0.$$

Proof. The terms in the above series can be grouped together by the value of $r_\alpha r_\beta$, where $r_\alpha r_\beta \neq e$. By Proposition 4.1, each group of terms is equal to zero. $\square$

The following theorem is presumably well-known when Σ is reduced, see for instance [16, 41], but since here Σ is possibly a non-reduced root system, for completeness we shall give the proof.

Theorem 4.3. *The Hamiltonian $\mathscr{H}$ is in a simple way conjugate to the radial part Δ_0*

$$\mathscr{H} = \zeta(q)^{\frac{1}{2}} \circ \Delta_0 \circ \zeta(q)^{-\frac{1}{2}},$$

where

$$\zeta(q) = \prod_{\alpha \in \Sigma^+} \langle q, \alpha \rangle^{m_\alpha}.$$

Here we choose the square root of $\zeta(q)$ by a fixed choice of $\sqrt{-1}$.

Proof. For $q \in \mathfrak{a}$

$$\zeta(q)^{-\frac{1}{2}} \circ \partial_q \circ \zeta(q)^{\frac{1}{2}} = \partial_q + \frac{1}{2} \partial_q (\log \zeta(q)),$$

and therefore

$$\zeta(q)^{-\frac{1}{2}} \circ \partial_q^2 \circ \zeta(q)^{\frac{1}{2}} = \partial_q^2 + \partial_q (\log \zeta(q)) \circ \partial_q + \zeta(q)^{-\frac{1}{2}} \partial_q^2 (\zeta(q)^{\frac{1}{2}}).$$

Moreover

$$\sum_{j=1}^n \partial_{q_j} (\log \zeta(q)) \partial_{q_j} = \sum_{\alpha > 0} \frac{m_\alpha}{\langle q, \alpha \rangle} \partial_\alpha,$$

and

$$\zeta(q)^{-\frac{1}{2}} \sum_{j=1}^n \partial_{q_j}^2 \big(\zeta(q)^{\frac{1}{2}} \big) = \sum_{\alpha > 0} \frac{m_\alpha}{2} \left(\frac{m_\alpha}{2} - 1 \right) \frac{\langle \alpha, \alpha \rangle}{\langle q, \alpha \rangle^2} + \frac{1}{4} \sum_{\substack{\alpha, \beta > 0 \\ \alpha \neq \beta}} m_\alpha m_\beta \frac{\langle \alpha, \beta \rangle}{\langle q, \alpha \rangle \langle q, \beta \rangle}. \tag{4.1}$$

If Σ is a reduced root system, then Corollary 4.2 finishes the proof in this case.

If Σ is an unreduced root system, then Σ is of type BC_n and the root systems B_n and C_n are contained in Σ. Since B_n and C_n are reduced root systems, using Corollary 4.2 we can write the second term of the right-hand side of (4.1) as

$$R(q) := \frac{1}{4} \sum_{\alpha,\beta\in\tilde{\Sigma}^+} m_\alpha m_{2\beta} \frac{\langle \alpha, \beta\rangle}{\langle q, \alpha\rangle\langle q, \beta\rangle},$$

where $\tilde{\Sigma}^+ = \{\alpha \in \Sigma^+ \mid 2\alpha \in \Sigma\}$. On the other-hand $\langle \alpha, \beta\rangle = \delta_{\alpha\beta}$ for $\alpha, \beta \in \tilde{\Sigma}^+$, and then the rational function $R(q)$ can be written as

$$R(q) = \frac{1}{2} \sum_{\alpha>0} m_\alpha m_{2\alpha} \frac{\langle \alpha, \alpha\rangle}{\langle q, \alpha\rangle^2}.$$

□

As a natural consequence of Theorem 3.4 and Theorem 4.3, we obtain the following corollary.

Corollary 4.4. *For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $q \in \mathfrak{p}$, $\Phi_\lambda(q) := \zeta(q)^{\frac{1}{2}}\psi(\lambda, q)$ is an eigenfunction for the Hamiltonian $\mathscr{H}$ with eigenvalue $-\langle\lambda, \lambda\rangle$.*

Example 4.5. In this example we consider one particle in the n-dimensional Euclidean space $\mathbb{R}^n$. In this example, the Hamiltonian $\mathscr{H}$ is given by

$$\mathscr{H} = \frac{d^2}{dq^2} - \frac{(n-1)(n-3)}{4q^2}.$$

It is clear that $\mathscr{H} q^{\frac{(n-1)}{2}} = 0$. Set $\Phi_\lambda(q) := q^{\frac{(n-1)}{2}}\psi(\lambda, q)$ to be an eigenfunction for $\mathscr{H}$ with eigenvalue $-k^2$, where $k \in \mathbb{R}$. Therefore, and after substituting Φ_λ, the operator $\mathscr{H}$ becomes the desired operator

$$\Delta_0 = \frac{d^2}{dq^2} + \frac{(n-1)}{q}\frac{d}{dq},$$

where $\psi(\lambda, q)$ is an eigenfunction for Δ_0 with eigenvalue $-k^2$. The explicit formula of $\psi(\lambda, q)$ is given below by Theorem 7.1 (when $G = SO_0(n, 1)$).

It is well known that the spherical functions play a fundamental role in Harmonic analysis, where the literature is very rich by its applications to analysis on semi-simple Lie groups. Except for the case of complex Lie groups, only in few examples we know an explicit formula of the spherical functions. In some interesting cases, for instance $SU^*(2n)/Sp(n)$ and $SU(p, q)/S(U(p)\times U(q))$, we are able to give an explicit formula for the spherical function $\psi(\lambda, X)$. In particular, these explicit formulas give concrete solutions for n-body problems, which are related to quantum mechanics. Other interesting cases are also investigated.

In the case of complex Lie groups, the integral representation of $\psi(\lambda, X)$ appears in Harish-Chandra's paper [20], where its explicit formula is given. Another proof can be found in Berline-Getzler-Vergne's book [2]. Our method for defining $\psi(\lambda, X)$ as a limit gives an alternative simple proof of the Harish-Chandra integral.

For symmetric spaces with root system of type A_{n-1} ($n = 2, 3, \ldots$), we use the generalized binomial formula proved in [35] in order to find the Taylor series of $\psi(\lambda, X)$.

5. Explicit formula when $\mathfrak{p} = \mathfrak{su}^*(2n) \cap iu(2n)$

Let $G = SU^*(2n)$ be given by

$$SU^*(2n) = \left\{ g \in SL(2n, \mathbb{C}) \;\mid\; g = J_n \bar{g} J_n^{-1} \right\},$$

where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Let $\mathfrak{su}^*(2n)$ be the Lie algebra of $G = SU^*(2n)$. With respect to the Cartan involution $\theta(X) = -J_n X^t J_n^{-1}$ $(X \in \mathfrak{su}^*(2n))$, we have $\mathfrak{su}^*(2n) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(2n) = \mathfrak{sp}(n), \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{su}^*(2n) \cap i\mathfrak{u}(2n).$$

The symmetric space $SU^*(2n)/Sp(n)$ is a semisimple symmetric space of type II in Cartan notation, and its curvature is negative.

For two vectors $k, x \in \mathbb{R}^n$, and for an integer $s \geq 2$, define

$$\begin{aligned} &D_s(k_1, \ldots, k_{s-1}; x_1, \ldots, x_s) \\ &= \exp\left(-\sum_{i=1}^{s-1} 2\coth(x_i - x_s) \frac{\partial}{\partial k_i} + \sum_{1 \leq i < j \leq s-1} 2\,\mathrm{sh}^{-2}(x_i - x_j) \frac{\partial^2}{\partial k_i \partial k_j} \right) \prod_{i=1}^{s-1} k_i. \end{aligned}$$

For $1 \leq i \leq s-1$, put $\partial_i - \partial_s := \dfrac{\partial}{\partial x_i} - \dfrac{\partial}{\partial x_s}$. Set

$$\mathbb{D}_s(\partial_1 - \partial_s, \ldots, \partial_{s-1} - \partial_s; x_1, \ldots, x_s)$$

to be the differential operator obtained after replacing k_i by[1] $\partial_i - \partial_s$ in the operator $D_s(k_1, \ldots, k_{s-1}; x_1, \ldots, x_s)$.

Let $X = \mathrm{diag}(x_1, \ldots, x_n; x_1, \ldots, x_n)$ be a diagonal matrix such that $\sum_{i=1}^n x_i = 0$, and let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. The spherical function $\varphi_\lambda(g)$ for the semisimple symmetric space $SU^*(2n)/Sp(n)$ is given in [7] by

$$\begin{aligned} \varphi_\lambda(\exp(X)) = \prod_{j=1}^n (2j-1)! \prod_{1 \leq j < \kappa \leq n} \Big((i\lambda_j - i\lambda_\kappa)^3 - 4(i\lambda_j - i\lambda_\kappa) \Big)^{-1} \\ \prod_{1 \leq j < \kappa \leq n} \mathrm{sh}^{-2}(x_j - x_\kappa) \sum_{\omega \in S_n} (-1)^\omega \Psi(i\omega(\lambda); x_1, \ldots, x_n), \end{aligned}$$

where, for two vectors $k, x \in \mathbb{R}^n$, the function $\Psi(k, x)$ is given by

$$\Psi(k, x) = \mathbb{D}_n \circ \mathbb{D}_{n-1} \circ \cdots \circ \mathbb{D}_2 \exp\Big(\sum_{i=1}^n k_i x_i \Big).$$

For $k, x \in \mathbb{R}^n$, define the following polynomial differential operator on k

$$\begin{aligned} &\widetilde{D}_s(k_1, \ldots, k_{s-1}; x_1, \ldots, x_s) \\ &= \exp\left(-\sum_{i=1}^{s-1} \frac{2}{(x_i - x_s)} \frac{\partial}{\partial k_i} + \sum_{1 \leq i < j \leq s-1} \frac{2}{(x_i - x_j)^2} \frac{\partial^2}{\partial k_i \partial k_j} \right) \prod_{i=1}^{s-1} k_i. \end{aligned}$$

Also, define $\widetilde{\mathbb{D}}_s$ to be the differential operator obtained by substitution of $\partial_i - \partial_s$ for k_i in $\widetilde{D}_s$.

[1] This is a small correction to [7], where the authors replace k_i by ∂_i.

Using Theorem 3.4, explicit formula for the spherical function $\psi(\lambda, X)$, for the flat symmetric space $\mathfrak{p} = \mathfrak{su}^*(2n) \cap i\mathfrak{u}(2n)$, holds.

Theorem 5.1. *Let* $X = \mathrm{diag}(x_1, \ldots, x_n; x_1, \ldots, x_n) \in \mathfrak{p}$, *and* $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n; \lambda_1, \ldots, \lambda_n) \in \mathfrak{a}_{\mathbb{C}}$. *The spherical function* ψ *is given by*

$$\psi(\lambda, X) = \prod_{j=1}^{n}(2j-1)! \prod_{1\le j<\kappa\le n} (x_j - x_\kappa)^{-2} \prod_{1\le j<\kappa\le n} (i\lambda_j - i\lambda_\kappa)^{-3}$$
$$\sum_{\omega\in S_n} (-1)^{\omega} \widetilde{\Psi}(i\lambda_{\omega(1)}, \ldots, i\lambda_{\omega(n)}; x_1, \ldots, x_n),$$

where

$$\widetilde{\Psi}(k, x) = \widetilde{\mathbb{D}}_n \circ \widetilde{\mathbb{D}}_{n-1} \circ \cdots \circ \widetilde{\mathbb{D}}_2 \exp\Big(\sum_{i=1}^{n} k_i x_i\Big).$$

In [3, 4] the authors were able to give an explicit formula for the spherical function $\psi(\lambda, X)$ only for $n = 2, 3$ and 4. Their strategy is to consider $\psi(\lambda, X)$ as an eigenfunction for the Laplacian operator. In order to check that our formula for $\psi(\lambda, X)$ agrees with [3, 4], we shall write Theorem 5.1 for $n = 3$. The case $n = 2$ is easy to check, and we leave the case $n = 4$ for the reader to verify.

Fix $n = 3$. In this case

$$\mathbb{D}_2(\partial_1 - \partial_2, x_1, x_2) = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) - 2\coth(x_1 - x_2).$$

The differential operator

$$\begin{aligned}
&\mathbb{D}_3(\partial_1 - \partial_3, \partial_2 - \partial_3, x_1, x_2, x_3) = \\
&= \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right)\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right) \\
&-2\coth(x_1 - x_3)\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\right) - 2\coth(x_2 - x_3)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right) \\
&+2\,\mathrm{sh}^{-2}(x_1 - x_2) + 4\coth(x_1 - x_3)\coth(x_2 - x_3).
\end{aligned}$$

Hence, the spherical function on the symmetric space $SU^*(6)/Sp(3)$ is given by

$$\varphi_\lambda(\exp(X)) = 3!5! \prod_{1\le j<\kappa\le 3} \mathrm{sh}^{-2}(x_j - x_\kappa) \prod_{1\le j<\kappa\le 3} \big((i\lambda_j - i\lambda_\kappa)^3 - 4(i\lambda_1 - i\lambda_j)\big)^{-1}$$
$$\sum_{\omega\in S_3} (-1)^{\omega} \Psi(i\lambda_{\omega(1)}, i\lambda_{\omega(2)}, i\lambda_{\omega(3)}; x_1, x_2, x_3),$$

where

$$\begin{aligned}\Psi(i\lambda_1,i\lambda_2,i\lambda_3;x_1,x_2,x_3) &= e^{i\big(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3\big)}\\ &\Big\{4\frac{\mathrm{ch}(x_1-x_2)}{\mathrm{sh}^3(x_1-x_2)}+4\frac{\coth(x_1-x_3)}{\mathrm{sh}^2(x_1-x_2)}\\ &-4\frac{\coth(x_2-x_3)}{\mathrm{sh}^2(x_1-x_2)}-4\frac{\coth(x_1-x_2)}{\mathrm{sh}^2(x_1-x_2)}\\ &-2(i\lambda_2-i\lambda_3)(i\lambda_1-i\lambda_3)\coth(x_1-x_2)\\ &-2(i\lambda_1-i\lambda_2)(i\lambda_2-i\lambda_3)\coth(x_1-x_3)\\ &-2(i\lambda_1-i\lambda_3)(i\lambda_1-i\lambda_2)\coth(x_2-x_3)\\ &+4(i\lambda_1-i\lambda_2)\coth(x_1-x_3)\coth(x_2-x_3)\\ &+4(i\lambda_2-i\lambda_3)\coth(x_1-x_3)\coth(x_1-x_2)\\ &+4(i\lambda_1-i\lambda_3)\coth(x_1-x_2)\coth(x_2-x_3)\\ &-8\coth(x_1-x_2)\coth(x_1-x_3)\coth(x_2-x_3)\\ &+\mathbb{V}(i\lambda_1,i\lambda_2,i\lambda_3)\Big\}.\end{aligned}$$

Here $\mathbb{V}(a,b,c)$ denotes the Vandermonde determinant. Defining

$$\tau_{j,\kappa}=(i\lambda_j-i\lambda_\kappa)(x_j-x_\kappa),$$

one obtains

$$\begin{aligned}\psi(\lambda,X) &= \lim_{\varepsilon\to 0}\varphi_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon X))\\ &= \frac{3!5!}{\prod_{1\le j<\kappa\le 3}(x_j-x_\kappa)^2(i\lambda_j-i\lambda_\kappa)^2}\\ &\quad\sum_{\omega\in S_3}(-1)^\omega\chi(i\lambda_{\omega(1)},i\lambda_{\omega(2)},i\lambda_{\omega(3)};x_1,x_2,x_3)e^{i\langle\omega(\lambda),X\rangle}\end{aligned}$$

where

$$\begin{aligned}\chi(i\lambda_1,i\lambda_2,i\lambda_3;x_1,x_2,x_3) &= 1-2\Big[\frac{1}{\tau_{1,2}}+\frac{1}{\tau_{1,3}}+\frac{1}{\tau_{2,3}}\Big]\\ &+4\Big[\frac{1}{\tau_{1,2}\tau_{1,3}}+\frac{1}{\tau_{1,2}\tau_{2,3}}+\frac{1}{\tau_{1,3}\tau_{2,3}}\Big]-\frac{12}{\tau_{1,2}\tau_{1,3}\tau_{2,3}}.\end{aligned}$$

It is remarkable that the series of $\psi(\lambda,X)$ stops at the order of the inverse of the Vandermonde determinant, which agrees with [3, 4].

Remark 5.2. (i) From the above computation, it follows that when $n=3$, the spherical function $\psi(\lambda,X)$ is a polynomial in $1/\tau_{j,\kappa}$, with $1\le j<\kappa\le 3$. Indeed this fact holds for all n, where the last term is given by $\prod_{1\le j<\kappa\le n}1/\tau_{j,\kappa}$, up to a constant. This follows from [5, Theorem 3.12], where we were able to prove a unified general formula for the spherical functions $\psi(\lambda,X)$, when the root system $\Sigma(\mathfrak{g},\mathfrak{a})$ is reduced and the multiplicities of the roots are even. Our formula is expressed in terms of a differential shift operator in $\mathbb{C}[\mathfrak{a}_\mathbb{C}]\otimes S(\mathfrak{a}_\mathbb{C})$. In particular, when $G/K=SU^*(2n)/Sp(n)$, where $m_\alpha=4$ for all α, it is easy to check the claim above from our general formula. However, in [5] we investigate a more general class of spherical functions, or Bessel functions, in the setting of special functions related to root systems.

(ii) In [8], the authors give also a general formula for $\psi(\lambda, X)$, which they called the Baker-Akhiezer function. Their formula, which is presented in a different way than ours in [5, Theorem 3.12], is expressed in terms of the Calogero-Moser operator applied $(\sum_{\alpha>0} m_\alpha)$-times to the function $\prod_{\alpha\in\Sigma^+}\langle\alpha, X\rangle^{m_\alpha} e^{\langle\lambda, X\rangle}$.

Next, we shall write the Calogero-Moser system in the quaternionic case. This result leads to other differential equations where the solution can be obtained explicitly from Theorem 5.1.

Let $X = (X_{i,j})$ be a $n\times n$ quaternionic matrix, and let $\mathscr{L}$ be the Laplacian operator $\mathscr{L} = -\frac{\partial^2}{\partial X_{i,j}^2}$. Its eigenfunctions are plane waves

$$\mathscr{L} e^{i\,\mathrm{tr}(\lambda X)} = \mathrm{tr}(\lambda^2) e^{i\,\mathrm{tr}(\lambda X)},$$

for a hermitian matrix $\lambda \in \mathrm{Herm}(n, \mathbb{H})$. A $Sp(n)$-invariant eigenfunction of $\mathscr{L}$, for the same energy $\mathrm{tr}(\lambda^2)$, can be constructed by the superposition

$$\psi(\lambda, X) = \int_{Sp(n)} e^{i\,\mathrm{tr}(\lambda u X u^{-1})} du.$$

This integral is a function of the n eigenvalues λ_i of λ and the n eigenvalues x_i of X. The Laplacian $\mathscr{L}$ can be written in terms of the eigenvalues x_i as a Schrödinger differential operator

$$\Big\{\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 4\sum_{i\neq j} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}\Big\}\psi(\lambda, X) = -\sum_{i=1}^n \lambda_i^2 \psi(\lambda, X). \tag{5.1}$$

Note that the equation (5.1) is not symmetric under interchange of the matrices λ and X, but the solution $\psi(\lambda, X)$ satisfies this symmetry as we can see from its integral representation (see also Remark 3.5(iii)).

Set

$$\Phi(\lambda, X) = \prod_{1\le i<j\le n} (x_i - x_j)^2 \psi(\lambda, X).$$

Thus, Φ satisfies the following Hamiltonian

$$\Big\{\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 4\sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\Big\}\Phi(\lambda, X) = -\sum_{i=1}^n \lambda_i^2 \Phi(\lambda, X). \tag{5.2}$$

This Schrödinger equation is a simple Calogero-Moser system. It is a n-body problem with a rational potential.

The solution of (5.2) is of the form

$$\Phi(\lambda, X) = e^{i\sum_{j=1}^n \lambda_j x_j}\Upsilon(\lambda, X),$$

where Υ is a solution of

$$\Big\{\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + 2i\sum_{j=1}^n \lambda_j \frac{\partial}{\partial x_j} - 4\sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\Big\}\Upsilon = 0.$$

Since $\Big\{\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} - 4\sum_{1\le i<j\le n} \frac{1}{(x_i - x_j)^2}\Big\}(x_i - x_j)^{-1} = 0$, the solution of (5.2) can be written as

$$\Phi(\lambda, X) = \prod_{1\le i<j\le n} (x_i - x_j)^{-1} f(\lambda, X),$$

where f is a polynomial of degree $n(n-1)/2$ in the variables $x_1, \ldots, x_n$, such that

$$\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2} + 2\sum_{j=1}^n \Big(\frac{\partial f}{\partial x_j} + i\lambda_j f\Big)\Big(\prod_{i<j}(x_i - x_j)\frac{\partial}{\partial x_j}\prod_{i<j}(x_i - x_j)^{-1}\Big) + 2i\sum_{j=1}^n \lambda_j \frac{\partial f}{\partial x_j} = 0.$$

As an immediate application of Theorem 5.1, one can obtain explicitly the solution f of the above differential equation. For $n = 2, 3$ and 4, the solution can also be found in [3, 4].

The investigation detailed above holds also for real and complex Hermitian matrices, where the Taylor series of the corresponding spherical functions $\psi(\lambda, X)$ will be given later in Section 10.

6. Explicit formula when $\mathfrak{p} \simeq M(p, q; \mathbb{C})$

For $q \geq p$, let $G = SU(p, q)$ be the group of all complex $(p+q) \times (p+q)$ matrices with determinant 1, which leave invariant the hermitian form

$$x_1\bar{x}_1 + \cdots + x_p\bar{x}_p - x_{p+1}\bar{x}_{p+1} - \cdots - x_{p+q}\bar{x}_{p+q}.$$

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{su}(p, q)$ is a real semisimple Lie algebra. With respect to the Cartan involution $\theta(X) = I_{p,q} X I_{p,q}$, the Lie algebra $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \;\middle|\; A \in \mathfrak{u}(p),\ B \in \mathfrak{u}(q),\ \mathrm{tr}(A + B) = 0 \right\}$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \;\middle|\; C \in M(p, q; \mathbb{C}) \right\}.$$

Here $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$. We may choose

$$\mathfrak{a} = \left\{ X = \begin{bmatrix} \mathbf{0}_{p,p} & X & \mathbf{0}_{p,q-p} \\ X & \mathbf{0}_{p,p} & \mathbf{0}_{p,q-p} \\ \mathbf{0}_{q-p,p} & \mathbf{0}_{q-p,p} & \mathbf{0}_{q-p,q-p} \end{bmatrix} \;\middle|\; X = \mathrm{diag}(x_1, \ldots, x_p),\ x_i \in \mathbb{R} \right\}$$

For $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathfrak{a}_{\mathbb{C}}^*$ and $X \in \mathfrak{a}$, the spherical function $\varphi_\lambda(g)$ on the symmetric space $SU(p, q)/S(U(p) \times U(q))$ is given in [25] by

$$\begin{aligned}\varphi_\lambda(\exp(X)) \;=\; & \frac{c_0}{\prod_{1\leq i<j\leq p}(\lambda_i^2 - \lambda_j^2)\prod_{1\leq i<j\leq p}\big(\mathrm{ch}(2x_i) - \mathrm{ch}(2x_j)\big)} \\ & \det\left[{}_2F_1\left(\frac{1+i\lambda_j}{2} + \frac{q-p}{2}, \frac{1-i\lambda_j}{2} + \frac{q-p}{2}, q-p+1; -\mathrm{sh}^2 x_k\right)\right]_{1\leq j,k\leq p},\end{aligned} \tag{6.1}$$

where

$$c_0 = (-1)^{\frac{p(p-1)}{2}} 2^{\frac{3p(p-1)}{2}} \prod_{j=1}^{p-1}(q - p + j)^{p-j} j!,$$

and ${}_2F_1$ is the Gaussian hypergeometric function.

Theorem 6.1. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and $X \in \mathfrak{p}$. The corresponding spherical function to the flat symmetric space $M(p,q;\mathbb{C})$, is given by*

$$\begin{aligned}\psi(\lambda, X) &= \int_{U(p)\times U(q)} e^{i\,\mathrm{tr}(\lambda u X u^{-1})} du \\ &= c_0' \prod_{j=1}^{p} (x_j\lambda_j)^{p-q} \frac{\det\Big[J_{q-p}(x_j\lambda_\kappa)\Big]_{1\le j,\kappa\le p}}{\prod_{1\le i<j\le p}(\lambda_i^2-\lambda_j^2)(x_i^2-x_j^2)},\end{aligned}$$

where $c_0' = 2^{q+p(p-1)}\Gamma(q-p+1)^p \prod_{j=1}^{p-1}(q-p+j)^{p-j} j!$, and J_ν is the Bessel function of the first kind.

Proof. For $\varepsilon > 0$, write $\varphi_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon X))$ as

$$\varphi_{\frac{\lambda}{\varepsilon}}(\exp(\varepsilon X)) = \frac{c_0 2^{\frac{p(1-p)}{2}}\ \varepsilon^{p(p-1)}}{\prod\limits_{1\le i<j\le p}(\lambda_i^2-\lambda_j^2)\prod\limits_{1\le i<j\le p}\big(\mathrm{sh}^2(\varepsilon x_i)-\mathrm{sh}^2(\varepsilon x_j)\big)}\times$$
$$\sum_{\omega\in S_n}(-1)^\omega\prod_{j=1}^{p} {}_2F_1\left(\frac{1+i\lambda_{\omega(j)}/\varepsilon}{2}+\frac{q-p}{2}, \frac{1-i\lambda_{\omega(j)}/\varepsilon}{2}+\frac{q-p}{2}, q-p+1; -\mathrm{sh}^2(\varepsilon x_j)\right).$$

Using the fact that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\Big\{1+\frac{1}{2z}(a-b)(a+b-1)+\mathscr{O}(z^{-2})\Big\}, \qquad \text{if } \ z\to\infty,$$

one can prove that

$$\begin{aligned}&\lim_{\varepsilon\to 0} {}_2F_1\left(\frac{1+i\lambda_{\omega(j)}/\varepsilon}{2}+\frac{q-p}{2}, \frac{1-i\lambda_{\omega(j)}/\varepsilon}{2}+\frac{q-p}{2}, q-p+1; -\mathrm{sh}^2(\varepsilon x_j)\right)\\ &= \Gamma(q-p+1)\left(\frac{x_j\lambda_{\omega(j)}}{2}\right)^{p-q} J_{q-p}(x_j\lambda_{\omega(j)}).\end{aligned}$$

Thus, the theorem holds. ☐

(After the work on this paper was completed, we learned that the explicit formula, presented above in Theorem 6.1, was proved earlier by Meaney [33] in connection with the inverse of the Abel transform for $G = SU(p,q)$. We thank M. Rösler for bringing the paper [33] to our attention.)

Recall that $\mathscr{D}(G_0/K)$, where $G_0 = K \ltimes \mathfrak{p}$, denotes the algebra of all G_0-invariant differential operators on G_0/K. Let $\delta(\mathscr{D}(G_0/K))$ be the algebra of all radial parts of invariant differential operators.

Using [1] and Theorem 3.3, the following proposition holds.

Proposition 6.2. *For $0 \le j \le p-1$, define*

$$\Delta_j := \omega^{-1}\mathbb{S}_{j+1}(\mathscr{L}_1,\dots,\mathscr{L}_p)\circ\omega,$$

where

$$\mathscr{L}_i = \frac{\partial^2}{\partial x_i^2} + \left(\frac{2(q-p)+1}{x_i}\right)\frac{\partial}{\partial x_i}, \qquad \omega(x) = 2^{p(p-1)}\prod_{1\le i<j\le p}(x_i^2-x_j^2),$$

and $\mathbb{S}_\kappa(\mathscr{L}_1, \dots, \mathscr{L}_p)$ *is the* κ*-th elementary symmetric polynomial in* $\mathscr{L}_1, \dots, \mathscr{L}_p$*. The operators* Δ_j*, with* $0 \le j \le p-1$*, form a system of generators for* $\delta(\mathscr{D}(G_0/K))$*. Moreover, for* $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p$*, we have*

$$\Delta_j \psi(\lambda, X) = \Upsilon_j(\lambda)\psi(\lambda, X)$$

for all $0 \le j \le p-1$*, where*

$$\Upsilon_j(\lambda) = \mathbb{S}_{j+1}(\lambda_1^2, \dots, \lambda_p^2).$$

Proof. For $0 \le j \le p-1$, let $\widetilde{\Delta}_j = \widetilde{\omega}^{-1}\mathbb{S}_{j+1}(\widetilde{\mathscr{L}_1}, \widetilde{\mathscr{L}_2}, \dots, \widetilde{\mathscr{L}_p}) \circ \widetilde{\omega}$, where $\widetilde{\omega}(\exp(X)) = 2^{\frac{p(p-1)}{2}} \prod_{1 \le i < j \le p} (\operatorname{ch} 2x_j - \operatorname{ch} 2x_j)$, for $X \in \mathfrak{a}$, and $\widetilde{\mathscr{L}_i} = \frac{\partial^2}{\partial x_i^2} + 2((q-p)\coth x_i + \coth 2x_i)\frac{\partial}{\partial x_i}$. By the second theorem of Berezin-Karpelevič [1], we know that the spherical functions $\varphi_\lambda(\exp(X))$ are eigenfunctions for $\widetilde{\Delta}_j$, $(j = 0, \dots, p-1)$, with respect to the eigenvalues $a_j(\lambda)$ given by $\prod_{i=1}^p \Big(\zeta - (\lambda_i^2 + (q-p+1)^2)\Big) = \sum_{j=0}^p a_{j-1}(\lambda)\zeta^{p-j}$. Using Theorem 3.3 and the fact that every differential operator $D \in \delta(\mathscr{D}(G/K))$ has $\varphi_\lambda(\exp(X))$ as eigenfunctions if and only if D can be written as a polynomial in the $\widetilde{\Delta}_j$, the proof is complete. $\square$

7. The real rank one case

Let G be a Lie group of real rank one. There are only four type of groups G with real rank one namely

$$SO_0(n,1), \quad SU(n,1), \quad Sp(n,1), \quad \text{and } F_{4(-20)}.$$

For the first three groups we will use the standard notation $SU(n,1;\mathbb{F})$ where $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Let $K = S(U(n,\mathbb{F}) \times U(1,\mathbb{F}))$ be a maximal compact subgroup in $SU(n,1;\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $K = SO(9)$ be a maximal abelian subgroup in $F_{4(-20)}$.

Fix a "normalized" element $L \in \mathfrak{p}$, and let $\mathfrak{a} = \mathbb{R}L$ be a maximal abelian subspace of $\mathfrak{p}$. Set $\alpha = 1$. The nonzero eigenvalues of $\operatorname{ad} L$ are $\pm\alpha$ if $G = SO_0(n,1)$, and $\pm\alpha$, $\pm 2\alpha$ if $G = SU(n,1)$, $Sp(n,1)$, $F_{4(-20)}$. Set $d = 1$ if $G = SO_0(n,1)$, $d = 2$ if $G = SU(n,1)$, $d = 4$ if $G = Sp(n,1)$, and $d = 8$ if $G = F_{4(-20)}$. The multiplicity of α is equal to $m_\alpha = d(n-1)$ for $G = SU(n,1;\mathbb{F})$, and $m_\alpha = d = 8$ for $G = F_{4(-20)}$. The multiplicity of the root 2α is equal to $m_{2\alpha} = d-1$.

For Lie groups of real rank one, the algebra $\mathscr{D}(G/K)$ of G-invariant differential operators on G/K is generated by the Laplace-Beltrami operator, where its radial part defines the differential equation

$$\left\{\frac{d^2}{dt^2} + (m_\alpha \coth t + 2m_{2\alpha} \coth 2t)\frac{d}{dt}\right\} y = -(\lambda^2 + \rho^2) y \qquad (t \in \mathbb{R}), \tag{7.1}$$

where $\lambda \in \mathbb{C}$ and $\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha})$. The spherical function

$$\varphi_\lambda(\exp(tL)) = {}_2F_1\left(\frac{i\lambda + \rho}{2}, \frac{-i\lambda + \rho}{2}; \frac{m_\alpha + m_{2\alpha} + 1}{2}; -\operatorname{sh}^2 t\right)$$

is the unique solution of (7.1) that satisfies $\varphi_\lambda(e) = 1$ in the unit e of G.

Since the spherical function is given in terms of the Gaussian hypergeometric function, one can reproduce the same argument used in the proof of Theorem 6.1 to obtain the following explicit formula.

Theorem 7.1. *For $\lambda \in \mathfrak{a}_{\mathbb{C}}$, and $X = tL \in \mathfrak{p}$ with $t \in \mathbb{R}$* **(insert a ",")**

$$\psi(\lambda, X) = \Gamma\left(\frac{m_\alpha + m_{2\alpha} + 1}{2}\right)\left(\frac{\lambda t}{2}\right)^{-\frac{m_\alpha + m_{2\alpha} - 1}{2}} J_{\frac{m_\alpha + m_{2\alpha} - 1}{2}}(\lambda t),$$

where J_ν is the Bessel function of the first kind.

Remark 7.2. The above theorem is known, see de Jeu's thesis [29], but the present approach is different.

8. The case of symmetric spaces of type B_2, C_2, and BC_2

We shall consider below only rank two symmetric spaces G/K of type B_2, C_2 and BC_2.

Let $\mathfrak{g}$ be the Lie algebra of G, and recall that $\Sigma^+ \subset \Sigma$ denotes the set of all positive restricted roots associated with $(\mathfrak{g}, \mathfrak{a})$. In the case where $\mathfrak{g}$ has the root system B_2, we have

$$\Sigma^+ = \{\alpha_1 \pm \alpha_2, \alpha_1, \alpha_2\}.$$

In the case where $\mathfrak{g}$ has the root system C_2, we have

$$\Sigma^+ = \{\alpha_1 \pm \alpha_2, 2\alpha_1, 2\alpha_2\}.$$

Finally, when $\mathfrak{g}$ has the root system BC_2, we have

$$\Sigma^+ = \{\alpha_1 \pm \alpha_2, \alpha_1, \alpha_2, 2\alpha_1, 2\alpha_2\}.$$

Let $m_1 = m_{\alpha_i}$, $m_2 = m_{2\alpha_i}$, and $m_3 = m_{\alpha_1 \pm \alpha_2}$. Bellow, we give the list of symmetric spaces with root systems of type B_2, C_2, and BC_2.

G/K	Σ	m_1	m_2	m_3
$SO_0(2,2)/SO(2) \times SO(2)$	Special B_2	0	0	1
$SO_0(2,q)/SO(2) \times SO(q),\ (2+q=2\ell)$	B_2	$2(\ell-2)$	0	1
$SO_0(2,q)/SO(2) \times SO(q),\ (2+q=2\ell+1)$	B_2	$2(\ell-2)+1$	0	1
$SU(2,q)/S(U(2) \times U(q))$	BC_2	$2(q-2)$	1	2
$SU(2,2)/S(U(2) \times U(2))$	C_2	0	1	2
$SO^*(8)/U(4)$	C_2	0	1	4
$SO^*(10)/U(5)$	BC_2	4	1	4
$Sp(2,\mathbb{R})/U(2)$	C_2	0	1	1
$Sp(2,2)/Sp(2) \times Sp(2)$	C_2	0	3	4
$Sp(2,q)/Sp(2) \times Sp(q),\ (2+q=\ell)$	BC_2	$4(\ell-4)$	3	4
$E_6/spin(10) \times \mathbb{T}$	BC_2	6	1	8

Set

$$D := \frac{\Delta_1 - \Delta_2}{\coth 2\alpha_1 - \coth 2\alpha_2}$$

where

$$\Delta_i = \partial_{\alpha_i}^2 + (m_3 \coth \alpha_i + 2m_2 \coth 2\alpha_i)\partial_{\alpha_i}, \qquad i = 1, 2.$$

For $(t_1,t_2)\in\mathbb{R}^2$, every element $H\in\mathfrak{a}$ will be represented as $H_{(t_1,t_2)}$. For $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2$, write

$$\begin{aligned}\varphi_\lambda\big(\exp H_{(t_1,t_2)}\big) = &\prod_{q=0}^{[m_3/2]-1}\Big\{\Big(\langle i\lambda,\alpha_1+\alpha_2\rangle^2+4q^2\Big)\Big(\langle i\lambda,\alpha_1-\alpha_2\rangle^2+4q^2\Big)\Big\}^{-1}\times\\ &D^{[m_3/2]}\Big(\sum_{\omega\in S_2}P^{(\alpha,\beta)}_{i\omega(\lambda_1)}(t_1)P^{(\alpha,\beta)}_{i\omega(\lambda_2)}(t_1)\Big)(\mathbf{inserta}",")\end{aligned} \tag{8.1}$$

where

$$P^{(\alpha,\beta)}_\nu(t) = {}_2F_1\left(\frac12\Big(\nu+\frac{m_1+2m_2}{2}\Big),\alpha+\beta+1-\frac12\Big(\nu+\frac{m_1+2m_2}{2}\Big);\alpha+1;-\operatorname{sh}^2 t\right)$$

is the normalized Legendre polynomial, and $[x]$ denotes the integral part of x. The function φ_λ is the spherical function associated with the following groups

$$\begin{array}{ll} G=SU(2,q), & \text{with}\quad \alpha=q-2,\ \beta=0;\\ G=SU(2,2), & \text{with}\quad \alpha=0,\ \beta=0;\\ G=Sp(2,2), & \text{with}\quad \alpha=1,\ \beta=1;\\ G=Sp(2,q), & \text{with}\quad \alpha=2(q-2)+1,\ \beta=1;\\ G=SO^*(8), & \text{with}\quad \alpha=0,\ \beta=0;\\ G=E_6, & \text{with}\quad \alpha=2,\ \beta=0. \end{array}$$

Remark 8.1. (i) For the above list of groups, we expressed the spherical functions in a unified new way.

(ii) Note that

$$\big[\Delta_1-\Delta_2\big]\Big\{\sum_{\omega\in S_2}P^{(\alpha,\beta)}_{i\omega(\lambda_1)}(t_1)P^{(\alpha,\beta)}_{i\omega(\lambda_2)}(t_2)\Big\} = -(\lambda_1^2-\lambda_2^2)\det\Big(P^{(\alpha,\beta)}_{i\lambda_j}(t_k)\Big)_{1\le j,k\le 2}.$$

In particular, we obtain formula (8.1), for $G=SU(2,q)$, as a special case of (6.1).

For the symmetric space $SO^*(10)/U(5)$, the spherical function is given by

$$\begin{aligned}\varphi_\lambda\big(\exp H_{(t_1,t_2)}\big) = &\prod_{j=1}^{2}(\lambda_j^2+1)^{-2}\prod_{q=0}^{1}\Big\{\Big(\langle i\lambda,\alpha_1+\alpha_2\rangle^2+4q^2\Big)\Big(\langle i\lambda,\alpha_1-\alpha_2\rangle^2+4q^2\Big)\Big\}^{-1}\times\\ &D_1D_2D^2\Big(\sum_{\omega\in S_2}P^{(0,0)}_{i\omega(\lambda_1)}(t_1)P^{(0,0)}_{i\omega(\lambda_2)}(t_1)\Big)\end{aligned}$$

where D_1 and D_2 are the well known Dunkl-Opdam type shifted differential operators

$$D_1=\frac{1}{16\operatorname{sh}2\alpha_1\operatorname{sh}2\alpha_2}\Big\{\partial_{2\alpha_1}\partial_{2\alpha_2}+8\Big(\coth(\alpha_1+\alpha_2)\partial_{\alpha_1+\alpha_2}-\coth(\alpha_1-\alpha_2)\partial_{\alpha_1-\alpha_2}\Big)\Big\}$$

and

$$D_2=\frac{1}{16}\Big\{\partial_{2\alpha_1}\partial_{2\alpha_2}+8\Big(\coth(\alpha_1+\alpha_2)\partial_{\alpha_1+\alpha_2}-\coth(\alpha_1-\alpha_2)\partial_{\alpha_1-\alpha_2}\Big)\Big\}.$$

For $Sp(2,\mathbb{R})/U(2)$, we can use [14, (1.4)] in order to write the spherical function as

$$\varphi_\lambda\big(\exp H_{(t_1,t_2)}\big) = c\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle\coth\Big(\frac{\pi\langle\alpha,i\lambda\rangle}{\langle\alpha,\alpha\rangle}\Big)\int_{GL(2,\mathbb{C})}\Phi_{2\lambda}\big(k\exp H_{(\frac{t_1}{2},\frac{t_2}{2})}\big)dk$$

where Φ_λ is the spherical function on $Sp(2,\mathbb{C})$.

Now, for the spherical functions associated to symmetric spaces with restricted root system of type B_2, we will adopt the same exchange between B_2 and C_2 root systems used by Debiard and Gaveau in [12].

Let $\mathfrak{g}_b$ be the Lie algebra of a Lie group G_b with restricted root system Σ_b of type B_2, and let $\mathfrak{g}_c$ be the Lie algebra of a Lie group G_c with restricted root system Σ_c of type C_2. Then

$$\Sigma_b^+ = \{e_1 \pm e_2, e_1, e_2\}, \quad \Sigma_c^+ = \{2f_1, 2f_2, f_1 \pm f_2\}.$$

We will identify an element $H_t \in \mathfrak{a}_b$, where $t = (t_1, t_2) \in \mathbb{R}^2$, with an element $H_T \in \mathfrak{a}_c$, where $T = (T_1, T_2) \in \mathbb{R}^2$. Then we have an isomorphism $\Psi : \mathfrak{a}_b \to \mathfrak{a}_c$ such that $t_1 + t_2 = 2T_1$, and $t_1 - t_2 = 2T_2$. The isomorphism Ψ transforms the B_2 root system into the C_2 root system, where

$$m_{f_1 \pm f_2} = m_{e_i}, \qquad m_{2f_i} = m_{e_1 \pm e_2}.$$

Moreover, if $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_b^*$ and $\Lambda = (\Lambda_1, \Lambda_2) \in \mathfrak{a}_c^*$, we have an isomorphism $\Psi^* : \mathfrak{a}_b^* \to \mathfrak{a}_c^*$ such that $\Lambda_1 = \lambda_1 + \lambda_2$ and $\Lambda_2 = \lambda_1 - \lambda_2$. The mapping Ψ and Ψ^* induces an isomorphism $\Phi : C^\infty(K_b \setminus G_b/K_b) \to C^\infty(K_c \setminus G_c/K_c)$ such that

$$\Phi\Big(\varphi_\lambda^b\big(\exp(H_t)\big)\Big) = \varphi_\Lambda^c\big(\exp(H_T)\big).$$

Here φ_λ^b (resp. φ_Λ^c) denotes the spherical function on the symmetric space G_b/K_b (resp. G_c/K_c).

For the symmetric space $G_b/K_b = SO(2, 2+2k)/SO(2) \times SO(2+2k)$, which is of type B_2, we have $\Sigma_b^+ = \{\, e_1, e_2, e_1 \pm e_2 \,\}$ with $m_{e_i} = 2k$ and $m_{e_1 \pm e_2} = 1$. Let G_c/K_c be the associated symmetric space with restricted root system $\Sigma_c^+ = \{\, 2f_1, 2f_2, f_1 \pm f_2 \,\}$. Hence $m_2 = m_{2f_i} = 1$ and $m_3 = m_{f_1 \pm f_2} = 2k$. Therefore, the spherical function on G_c/K_c is given by

$$\begin{aligned}\varphi_\Lambda\big(\exp(H_T)\big) = &\prod_{q=0}^{[m_3/2]-1} \Big\{\Big(\langle i\Lambda, f_1 + f_2\rangle^2 + 4q^2\Big)\Big(\langle i\Lambda, f_1 - f_2\rangle^2 + 4q^2\Big)\Big\}^{-1} \\ &D^{[m_3/2]}\Big(\sum_{\omega \in S_2} P^{(0,0)}_{i\omega(\Lambda_1)}(T_1) P^{(0,0)}_{i\omega(\Lambda_2)}(T_2)\Big),\end{aligned}$$

and

$$\varphi_\lambda\big(\exp(H_t)\big) = \Phi^{-1}\Big(\varphi_\Lambda\big(\exp(H_T)\big)\Big),$$

is the spherical function on $SO(2, 2+2k)/SO(2) \times SO(2+2k)$.

Since the restricted root system associated with $SO_0(2,2)/SO(2) \times SO(2)$ can be seen as a special B_2 type, we reproduce the same argument used above to prove that the spherical function on $SO_0(2,2)/SO(2) \times SO(2)$ is given by

$$\varphi_\lambda\big(\exp(H_{(t_1,t_2)})\big) = P^{(0,0)}_{i(\lambda_1+\lambda_2)}\Big(\frac{t_1+t_2}{2}\Big) P^{(0,0)}_{i(\lambda_1-\lambda_2)}\Big(\frac{t_1-t_2}{2}\Big).$$

For the symmetric space $G_b/K_b = SO(2, 3+2k)/SO(2) \times SO(3+2k)$ we have $m_{e_i} = 2k+1$ and $m_{e_1 \pm e_2} = 1$. The associated symmetric space G_c/K_c has the following root system $\Sigma_c^+ = \{2f_1, 2f_2, f_1 \pm f_2\}$, where $m_2 = m_{2f_i} = 1$ and $m_3 =$

$m_{f_1 \pm f_2} = 2k+1$. Therefore, the spherical function on G_c/K_c is given by

$$\begin{aligned}\varphi_\Lambda\big(\exp(H_T)\big) &= c \prod_{q=0}^{[m_3/2]-1} \Big\{\Big(\langle i\Lambda, f_1+f_2\rangle^2 + (2q+1)^2\Big)\Big(\langle i\Lambda, f_1-f_2\rangle^2 + (2q+1)^2\Big)\Big\}^{-1} \\ &\quad \prod_{\beta\in\Sigma_c^+} \langle i\Lambda,\beta\rangle \coth\Big(\frac{\pi\langle i\Lambda,\beta\rangle}{\langle\beta,\beta\rangle}\Big) D^{[m_3/2]}\Big(\int_{GL(2,\mathbb{C})} \Phi_{2\Lambda}(k\exp(H_{T/2}))dk\Big),\end{aligned}$$

where Φ_Λ is the spherical function on the complex symmetric space $Sp(2,\mathbb{C})/USp(2)$.

Remark 8.2. By [14], up to a constant

$$\prod_{\beta\in\Sigma_c^+} \langle i\Lambda,\beta\rangle \coth\Big(\frac{\pi\langle i\Lambda,\beta\rangle}{\langle\beta,\beta\rangle}\Big) \int_{GL(2,\mathbb{C})} \Phi_{2\Lambda}\big(k\exp(H_{T/2})\big)dk,$$

is the spherical function on $Sp(2,\mathbb{R})/U(2)$.

Define the following differential operators

$$\widetilde{\Delta}_i = \partial^2_{\alpha_i} + \Big(\frac{m_{\alpha_1\pm\alpha_2}}{\alpha_i} + \frac{2m_{2\alpha_i}}{2\alpha_i}\Big)\partial_{\alpha_i}, \qquad i=1,2$$

and set

$$\widetilde{D} = \frac{\widetilde{\Delta}_1 - \widetilde{\Delta}_2}{2\alpha_1 - 2\alpha_2}.$$

Theorem 8.3. *Let $\lambda = (\lambda_1,\lambda_2)\in\mathbb{C}^2$, and $(t_1,t_2)\in\mathbb{R}^2$.*

(i) *When G/K is a symmetric space with root system of type C_2 or BC_2, except the cases $SO^*(10)/U(5)$ and $Sp(2,\mathbb{R})/U(2)$, the spherical function $\psi(\lambda,X)$ on the flat symmetric space $\mathfrak{p}$ is given by*

$$\begin{aligned}\psi(\lambda_1,\lambda_2;t_1,t_2) &= 2^{2\alpha}\Gamma(\alpha+1)^2\Big(\langle\lambda,\alpha_1+\alpha_2\rangle^2\langle\lambda,\alpha_1-\alpha_2\rangle^2\Big)^{-[m_3/2]} \\ &\quad \widetilde{D}^{[m_3/2]}\Big[\sum_{\omega\in S_2}(t_1\lambda_{\omega(1)})^{-\alpha}(t_2\lambda_{\omega(2)})^{-\alpha} J_\alpha(t_1\lambda_{\omega(1)})J_\alpha(t_2\lambda_{\omega(2)})\Big],\end{aligned}$$

where J_ν is the Bessel function of the first kind.

(ii) *For the flat symmetric space $\mathfrak{p}$ associated with $SO^*(10)/U(5)$, the spherical function $\psi(\lambda,X)$ is given by*

$$\begin{aligned}\psi(\lambda_1,\lambda_2;t_1,t_2) &= \Big(\prod_{j=1}^2 \lambda_j^{-4}\Big)\Big(\langle\lambda,\alpha_1+\alpha_2\rangle^2\langle\lambda,\alpha_1-\alpha_2\rangle^2\Big)^{-2} \\ &\quad \widetilde{D}_1\widetilde{D}_2\widetilde{D}^2\Big[\sum_{\omega\in S_2} J_0(t_1\lambda_{\omega(1)})J_0(t_2\lambda_{\omega(2)})\Big],\end{aligned}$$

where

$$\widetilde{D}_1 = \frac{1}{16(2\alpha_1)(2\alpha_2)}\Big(\partial_{2\alpha_1}\partial_{2\alpha_2} + 8\Big\{\frac{\partial_{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2} - \frac{\partial_{\alpha_1-\alpha_2}}{\alpha_1-\alpha_2}\Big\}\Big),$$

and

$$\widetilde{D}_2 = \frac{1}{16}\Big(\partial_{2\alpha_1}\partial_{2\alpha_2} + 8\Big\{\frac{\partial_{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2} - \frac{\partial_{\alpha_1-\alpha_2}}{\alpha_1-\alpha_2}\Big\}\Big).$$

(iii) *For the symmetric space $Sp(2,\mathbb{R})/U(2)$,*

$$\psi(\lambda_1,\lambda_2;t_1,t_2) = c\lim_{\varepsilon\to 0}\prod_{\alpha\in\Sigma^+}\left\langle \alpha,\frac{\lambda}{\varepsilon}\right\rangle\int_{GL(2,\mathbb{C})}\Phi_{2\frac{\lambda}{\varepsilon}}\big(k\exp(H_{(\varepsilon t_1/2,\varepsilon t_2/2)})\big)dk,$$

where Φ_λ is the spherical function on $Sp(2,\mathbb{C})$.

(iv) *For $G/K = SO_0(2,2)/SO(2)\times SO(2)$,*

$$\psi(\lambda_1,\lambda_2;t_1,t_2) = J_0\Big(\frac{(t_1+t_2)(\lambda_1+\lambda_2)}{2}\Big)J_0\Big(\frac{(t_1-t_2)(\lambda_1-\lambda_2)}{2}\Big).$$

(v) *For the symmetric space $G/K = SO(2,2+2k)/SO(2)\times SO(2+2k)$, the spherical function $\psi(\lambda,X)$ is given by*

$$\psi(\lambda_1,\lambda_2;t_1,t_2) = \Phi^{-1}(\psi_e(\Lambda_1,\Lambda_2;T_1,T_2)),$$

where

$$\psi_e(\Lambda_1,\Lambda_2;T_1,T_2) = \Big(\langle\Lambda,f_1+f_2\rangle^2\langle\Lambda,f_1-f_2\rangle^2\Big)^{-k}\widetilde{D}^k\Big[\sum_{\omega\in S_2}J_0(T_1\Lambda_{\omega(1)})J_0(T_2\Lambda_{\omega(2)})\Big],$$

with $t_1+t_2 = 2T_1$, $t_1-t_2 = 2T_2$, $\lambda_1+\lambda_2 = \Lambda_1$, and $\lambda_1-\lambda_2 = \Lambda_2$.

(vi) *For the symmetric space $SO(2,3+2k)/SO(2)\times SO(3+2k)$, the spherical function $\psi(\lambda,X)$ is given by*

$$\psi(\lambda_1,\lambda_2;t_1,t_2) = \Phi^{-1}(\psi_o(\Lambda_1,\Lambda_2;T_1,T_2)),$$

where

$$\psi_o(\Lambda_1,\Lambda_2;T_1,T_2) = \Big(\langle\Lambda,f_1+f_2\rangle^2\langle\Lambda,f_1-f_2\rangle^2\Big)^{-k}\widetilde{D}^k\Big[\psi_{Sp(2,\mathbb{R})/U(2)}(\Lambda;T_1,T_2)\Big]$$

and $\psi_{Sp(2,\mathbb{R})/U(2)}$ is the spherical function given in (iii).

9. The complex case

In the case when G is complex, the spherical function φ_λ, and consequently $\psi(\lambda,X)$ has an explicit formula.

As before, let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. Fix a Weyl positive chamber $\mathfrak{a}^+\subset\mathfrak{a}$, and let W be the Weyl group associated with this choice. Recall that ρ denotes the half-sum of positive roots with multiplicity counted. For $X\in\mathfrak{a}$, the spherical function on the complex group G is given in [18] by

$$\varphi_\lambda\big(\exp(X)\big) = \frac{\prod_{\alpha\in\Sigma^+}\langle\alpha,\rho\rangle\sum_{\omega\in W}(\det\omega)e^{\langle i\omega\lambda,X\rangle}}{\prod_{\alpha\in\Sigma^+}\langle\alpha,i\lambda\rangle\sum_{\omega\in W}(\det\omega)e^{\langle\omega\rho,X\rangle}}.$$

Note that

$$\sum_{\omega\in W}(\det\omega)e^{\langle\omega\rho,X\rangle} = e^{\langle\rho,X\rangle}\prod_{\alpha\in\Sigma^+}(1-e^{-2\langle\alpha,X\rangle}).$$

Using theorem 3.4 we obtain an explicit formula for the spherical functions on the tangent space of G/K at the origin, when G is complex.

Theorem 9.1. *For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $X \in \mathfrak{p}$*

$$\psi(\lambda, X) = \frac{\prod_{\alpha\in\Sigma^+} \langle \alpha, \rho\rangle}{\prod_{\alpha\in\Sigma^+} \langle \alpha, i\lambda\rangle \prod_{\alpha\in\Sigma^+} 2\langle \alpha, X\rangle} \sum_{\omega\in W} (\det\omega) e^{\langle i\omega\lambda, X\rangle}.$$

Recall that the integral representation of $\psi(\lambda, X)$ is given by

$$\psi(\lambda, X) = \int_K e^{iB(A_\lambda, \mathrm{Ad}(k)X)} dk.$$

When G is complex, this integral is the so-called Harish-Chandra integral. The above theorem is also proved by Harish-Chandra in [20] using other techniques.

10. Symmetric space with root system of the type A_{N-1}

In this section we will consider the following noncompact symmetric spaces with root system of the type A_{N-1} ($N = 2, 3, \ldots$)

$$GL(N, \mathbb{R})/O(N),\ GL(N, \mathbb{C})/U(N),\ GL(N, \mathbb{H})/Sp(N),$$

$$E_{6(-26)}/F_4,\ O(1, N)/O(N).$$

Let $\wp$ be a strictly positive parameter. Let $P_N = \mathbb{C}[x_1, \ldots, x_N]$ be the polynomial algebra in N independent variables, and $\Lambda_N \subset P_N$ be the algebra of symmetric polynomials. A partition is any sequence $\lambda = (\lambda_1, \ldots, \lambda_N, \ldots)$ of nonnegative integers in decreasing order $\lambda_1 \geq \cdots \geq \lambda_N \geq \cdots$ containing only finitely many nonzero terms. The number of nonzero terms in λ is the length of λ denoted by $l(\lambda)$. The sum $|\lambda| = \lambda_1 + \cdots + \lambda_N + \cdots$ is called the weight of λ. The set of partitions of weight N is denoted by $\mathscr{P}_N$. On this set there is a natural involution which, in the standard diagrammatic representation, corresponds to the transposition (reflection in the main diagonal). The image of a partition λ under this involution is called the conjugate of λ, and is denoted by λ'.

An important example of symmetric functions are the Jack polynomials. We give here their definition. Recall that on the set of partitions $\mathscr{P}_N$ there is the following dominance partial ordering: we write $\mu \leq \lambda$ if for all $i \geq 1$

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

Consider the following Calogero-Moser-Sutherland operator

$$\begin{aligned}\Delta_\wp^N &= \sum_{i=1}^N \Big(x_i \frac{\partial}{\partial x_i}\Big)^2 + \wp \sum_{1\leq i<j\leq N} \frac{x_i + x_j}{x_i - x_j}\Big(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\Big) - \wp(N-1) \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^N \Big(x_i \frac{\partial}{\partial x_i}\Big)^2 + 2\wp \sum_{i\neq j} \frac{x_i x_j}{x_i - x_j} \frac{\partial}{\partial x_i}.\end{aligned}$$

If $\wp$ is not a negative rational number or zero, then for any partition λ, such that $l(\lambda) \geq N$, there is a unique polynomial $P_\lambda(x, \wp) \in \Lambda_N$, called the Jack polynomial, such that

(i) $P_\lambda(x, \wp)$ is an eigenfunction of the $\Delta_\wp^N$ operator.

(ii) $P_\lambda(x, \wp) = m_\lambda + \sum_{\mu<\lambda} v_{\lambda,\mu} m_\mu$, where $v_{\lambda,\mu} \in \mathbb{C}$ and m_μ is the elementary symmetric polynomial.

Now we discuss the so-called shifted Jack polynomials investigated recently by Knop, Sahi, Okounkov and Olshanski (cf. [30] [35]). Let us denote by $\Lambda_{\wp,N}$ the algebra of polynomials $f(x_1, \dots, x_N)$ which are symmetric in the shifted variables $x_i\wp(1-i)$. Let us introduce the following function on the set of partition

$$H(\lambda, \wp) = \prod_{\square\in\lambda} (c_\wp(\square) + 1),$$

where

$$c_\wp(\square) = \lambda_i - j + \wp(\lambda'_j - i).$$

Here we identify a partition λ with its diagram

$$\lambda = \{ \square = (i,j) : 1 \le i \le l(\lambda),\ 1 \le j \le \lambda_i \}.$$

Let λ be a partition with $\lambda_{N+1} = 0$. There exists a unique shifted symmetric polynomial $P^*_\lambda(x, \wp) \in \Lambda_{N,\wp}$, called the shifted Jack polynomial, such that $\mathrm{degree}(P^*_\lambda) \le |\lambda|$, and

$$P^*_\lambda(\mu, \wp) = \begin{cases} H(\lambda, \wp), & \mu = \lambda \\ 0, & |\mu| \le |\lambda|,\ \ \mu \ne \lambda,\ \ \mu_{N+1} = 0. \end{cases}$$

Knop and Sahi proved that the shifted Jack polynomial $P^*_\lambda(x, \wp)$ satisfies the extra vanishing property $P^*_\lambda(\mu, \wp) = 0$ unless the diagram of μ is a subset of the diagram of λ, i.e. $\mu_i \le \lambda_i$ for all $i \ge 1$, and that $P^*_\lambda(x, \wp)$ is the usual Jack polynomial $P_\lambda(x, \wp)$ plus lower order terms. We shall write $\mu \subset \lambda$ to mean that the diagram of λ contains the diagram of μ.

By [32] and [43], we have the following branching rule for the Jack polynomials

$$P_\lambda(x_1, x_2, \dots, x_N, \wp) = \sum_{\mu\prec\lambda} \varrho_{\lambda/\mu}(\wp) x_1^{|\lambda/\mu|} P_\mu(x_2, \dots, x_N; \wp), \tag{10.1}$$

where $\mu \prec \lambda$ stands for the inequalities of interlacing

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \mu_{N-1} \ge \lambda_N,$$

the coefficient $\varrho_{\lambda/\mu}(\wp)$ is given by

$$\varrho_{\lambda/\mu}(\wp) = \prod_{1\le i\le j\le N-1} \frac{(\mu_i - \mu_j + \wp(j-i) + \wp)_{\mu_j - \lambda_{j+1}}}{(\mu_i - \mu_j + \wp(j-i) + 1)_{\mu_j - \lambda_{j+1}}} \frac{(\lambda_i - \mu_j + \wp(j-i) + 1)_{\mu_j - \lambda_{j+1}}}{(\lambda_i - \mu_j + \wp(j-i) + \wp)_{\mu_j - \lambda_{j+1}}},$$

and $|\lambda/\mu|$ denotes the weight of the skew diagram λ/μ which is equal to $|\lambda| - |\mu|$.

For the shifted Jack polynomial P^*_λ, Okounkov proved in [34] the following formula

$$P^*_\lambda(x_1, \dots, x_N; \wp) = \sum_{\mu\prec\lambda} \varrho_{\lambda/\mu}(\wp) \prod_{\square\in\lambda/\mu} (x_1 - c'_\wp(\square)) P^*_\mu(x_2, \dots, x_N; \wp), \tag{10.2}$$

where $\varrho_{\lambda/\mu}(\wp)$ is the same as for P_λ, and

$$c'_\wp(\square) = (j-1) - \wp(i-1), \qquad \text{for } \square = (i,j).$$

Let

$$H'(\lambda, \wp) = \prod_{\square\in\lambda} (c'_\wp(\square) + \wp).$$

The following was conjectured by Macdonald and proved by Stanley [43]

$$P_\lambda(1, \dots, 1; \wp) = \frac{(N\wp)_\lambda}{H'(\lambda, \wp)}$$

where

$$(N\wp)_\lambda = \prod_{\square\in\lambda}(N\wp + c'_\wp(\square)).$$

Next, we review the spherical functions on symmetric cones. For details, we refer to Faraut-Korányi's book [15, Chapter XI].

Let Ω be an open and convex cone associated with an Euclidean Jordan algebra $\mathbb{V}$. The cone Ω can be identified with one of the Riemannian symmetric spaces G/K listed in the beginning of this section.

Assume that $\mathbb{V}$ is a simple Euclidean Jordan algebra, i.e. $\mathbb{V}$ does not contain non-trivial ideals. Let N be the rank of $\mathbb{V}$, and let $\{c_1,\dots,c_N\}$ be a complete system of orthogonal idempotent elements. Each element x in $\mathbb{V}$ can be written as $x = k\sum_{j=1}^N x_jc_j$, with $k\in K$ and $x_j\in\mathbb{R}$.

For $\mathbf{m}=(m_1,\dots,m_N)$ and $x=\sum_{j=1}^N x_jc_j$, the spherical functions on Ω are given by

$$\varphi_{\mathbf{m}}(x) = \int_K \Delta_1^{m_1-m_2}(kx)\cdots\Delta_{n-1}^{m_{n-1}-m_n}(kx)\Delta_n^{m_n}(kx)dk,$$

where $\Delta_j(y)$ is the principal minor of order j of y, and dk denotes the normalized Haar measure on K. This formula corresponds to the classical Harish-Chandra formula for the spherical functions on G/K with $\mathbf{m}=\frac{\lambda+\rho}{2}$ where $\rho=(\rho_1,\dots,\rho_N)$, $\rho_j=\frac{1}{2}(2j-N-1)$, and $\lambda\in\mathbb{C}^N$.

If $\mathbf{m}=(m_1,\dots,m_N)\in\mathbb{N}^N$ such that $m_1\geq\cdots\geq m_N\geq 0$, the spherical function $\varphi_{\mathbf{m}}$ is a polynomial, and can be written in terms of the Jack polynomials. If $x = k\sum_{j=1}^N x_jc_j$

$$\varphi_{\mathbf{m}}(x) = \frac{P_{\mathbf{m}}(x_1,\dots,x_N;\wp)}{P_{\mathbf{m}}(1,\dots,1;\wp)}, \qquad \text{with} \quad \wp = \frac{2}{d},$$

where $d=1$ for $GL(N,\mathbb{R})/O(N)$, $d=2$ for $GL(N,\mathbb{C})/U(N)$, $d=4$ for $GL(N,\mathbb{H})/Sp(N)$, $d=8$ for $E_{6(-26)}/F_4$, and $d=N$ for $O(1,N)/O(N)$.

By [35, (2.6)], we have the following binomial formula

$$\begin{aligned}\frac{P_{\mathbf{m}}(1+x_1,\dots,1+x_N;\wp)}{P_{\mathbf{m}}(1,\dots,1;\wp)} &= \sum_{\mu\subset\mathbf{m}}\frac{P^*_\mu(\mathbf{m},\wp)P_\mu(x,\wp)}{P_\mu(1,\dots,1;\wp)H(\mu,\wp)} \\ &= \sum_{\mu\subset\mathbf{m}}\frac{P_\mu(\mathbf{m},\wp)P^*_\mu(x,\wp)}{P_\mu(1,\dots,1;\wp)H(\mu,\wp)}.\end{aligned} \tag{10.3}$$

For $\wp=1$, i.e. $d=2$, formula (10.3) reduces to the usual binomial formula

$$\frac{S_{\mathbf{m}}(1+x_1,\dots,1+x_N)}{S_{\mathbf{m}}(1,\dots,1)} = \sum_{\mu\subset\mathbf{m}}\frac{S^*_\mu(\mathbf{m})S_\mu(x)}{\prod_{\square=(i,j)\in\mu}(N+j-i)}$$

where S_μ is the Schur function

$$S_\mu(x_1,\dots,x_N) = \frac{\det\left(x_i^{\mu_j+N-j}\right)_{1\leq i,j\leq N}}{\prod_{1\leq i<j\leq N}(x_i-x_j)}$$

and

$$S^*_\mu(x) = \frac{\det\Big((x_i+N-i)\cdots(x_i-i+j-\mu_j+1)\Big)_{1\leq i,j\leq N}}{\prod_{1\leq i<j\leq N}(x_i-i-x_j+j)}.$$

See for instance [31].

Henceforth, ς denotes a very large integer. By using formula (10.2) N-times, we obtain

$$\begin{aligned}
&P^*_\mu(m_1\varsigma,\ldots,m_N\varsigma;\wp)\\
&=\sum_{\eta_1\prec\mu}\varrho_{\mu/\eta_1}\prod_{\square\in\mu/\eta_1}(m_1\varsigma-c_\wp(\square))\sum_{\eta_2\prec\eta_1}\varrho_{\eta_1/\eta_2}\prod_{\square\in\eta_1/\eta_2}(m_2\varsigma-c_\wp(\square))\\
&\cdots\sum_{\eta_{N-1}\prec\eta_{N-2}}\varrho_{\eta_{N-2}/\eta_{N-1}}\prod_{\square\in\eta_{N-2}/\eta_{N-1}}(m_{N-1}\varsigma-c_\wp(\square))P^*_{\eta_{N-1}}(m_N\varsigma,\wp)\\
&=\varsigma^{|\mu|}\sum_{\eta_1\prec\mu}\varrho_{\mu/\eta_1}\prod_{\square\in\mu/\eta_1}(m_1-\varsigma^{-1}c_\wp(\square))\sum_{\eta_2\prec\eta_1}\varrho_{\eta_1/\eta_2}\prod_{\square\in\eta_1/\eta_2}(m_2-\varsigma^{-1}c_\wp(\square))\\
&\cdots\sum_{\eta_{N-1}\prec\eta_{N-2}}\varrho_{\eta_{N-2}/\eta_{N-1}}\prod_{\square\in\eta_{N-2}/\eta_{N-1}}(m_{N-1}-\varsigma^{-1}c_\wp(\square))P^*_{\eta_{N-1}}(m_N,\wp)\\
&\sim\varsigma^{|\mu|}P_\mu(m_1,\ldots,m_N;\wp),\qquad \textit{when}\ \ \varsigma\to\infty.
\end{aligned}$$

For the Jack polynomial P_μ, we use formula (10.1) N-times to write

$$\begin{aligned}
&P_\mu(1-e^{\varsigma^{-1}s_1},\ldots,1-e^{\varsigma^{-1}s_N};\wp)\\
&=\sum_{\eta_1\prec\mu}\varrho_{\mu/\eta_1}(1-e^{\varsigma^{-1}s_1})^{|\mu/\eta_1|}\sum_{\eta_2\prec\eta_1}\varrho_{\eta_1/\eta_2}(1-e^{\varsigma^{-1}s_2})^{|\eta_1/\eta_2|}\\
&\cdots\sum_{\eta_{N-1}\prec\eta_{n-2}}\varrho_{\eta_{N-2}/\eta_{N-1}}(1-e^{\varsigma^{-1}s_{N-1}})^{|\eta_{N-2}/\eta_{N-1}|}P_{\eta_{N-1}}(1-e^{\varsigma^{-1}s_N},\wp)\\
&\sim\varsigma^{-|\mu|}\sum_{\eta_1\prec\mu}\varrho_{\mu/\eta_1}s_1^{|\mu/\eta_1|}\sum_{\eta_2\prec\eta_1}\varrho_{\eta_1/\eta_2}s_2^{|\eta_1/\eta_2|}\\
&\cdots\sum_{\eta_{N-1}\prec\eta_{N-2}}\varrho_{\eta_{N-2}/\eta_{N-1}}s_{N-1}^{|\eta_{N-2}/\eta_{N-1}|}P_{\eta_{N-1}}(s_N,\wp)\qquad \textit{when}\ \ \varsigma\to\infty\\
&=\varsigma^{-|\mu|}P_\mu(s_1,\ldots,s_N;\wp).
\end{aligned}$$

Thus, the following theorem holds

Theorem 10.1. *Let $X=\sum_{j=1}^N x_jc_j$, $\varsigma\in\mathbb{N}$, and $\mathbf{m}=(m_1,m_2,\ldots,m_N)\in\mathbb{N}^N$ such that $m_1\geq\cdots\geq m_N$. The following Taylor series holds*

$$\lim_{\varsigma\to\infty}\varphi_{\mathbf{m}\varsigma}\big(\exp(\varsigma^{-1}X)\big)=\sum_{\mu\in\mathscr{P}_N}\frac{P_\mu(m_1,\ldots,m_N;\frac{2}{d})P_\mu(x_1,\ldots,x_N;\frac{2}{d})}{P_\mu(1,\ldots,1;\frac{2}{d})H(\mu;\frac{2}{d})}.$$

Remark 10.2. (i) For a real vector $\nu=(\nu_1,\ldots,\nu_N)$ such that $\nu_1\geq\cdots\geq\nu_N$, write

$$\psi_c(\nu,X):=\int_K e^{B(A_\nu,\mathrm{Ad}(k)X)}dk.$$

As in Theorem 3.4, one can prove that

$$\begin{aligned}
\psi_c(\nu,X)&=\lim_{\varsigma\to\infty}\varphi_{[\varsigma\nu]}\big(\exp(\varsigma^{-1}X)\big)\\
&=\sum_{\mu\in\mathscr{P}_N}\frac{P_\mu(\nu_1,\ldots,\nu_N;\frac{2}{d})P_\mu(x_1,\ldots,x_N;\frac{2}{d})}{P_\mu(1,\ldots,1;\frac{2}{d})H(\mu;\frac{2}{d})},\qquad\text{(by Theorem 10.1)}
\end{aligned}$$

where $[\varsigma\nu]=([\varsigma\nu_1],\ldots,[\varsigma\nu_N])$ is the N-vector of integral parts.

(ii) Using [42], one can prove that for $\mathbf{m} = (m_1, m_2)$ **(insert a ",")**

$$\varphi_{\mathbf{m}}(x_1, x_2) = e^{m_1 x_1} e^{m_2 x_2} {}_2F_1\left(m_2 - m_1, \frac{d}{2}; d; 1 - e^{x_2 - x_1}\right).$$

In particular, we deduce that

$$\psi_c(\nu_1, \nu_2; x_1, x_2) = e^{\nu_1 x_1} e^{\nu_2 x_2} {}_1F_1\left(\frac{d}{2}, d; (\nu_1 - \nu_2)(x_1 - x_2)\right),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

(iii) The various Bessel functions $\psi(\lambda, X)$ and $\psi_c(\nu, X)$ coming from symmetric spaces of negative and positive curvature, respectively, are related to each other by the transformation $\psi(\lambda, X) = \psi_c(\nu, X)\,|_{\nu = i\lambda}$. This statement holds for every flat symmetric space.

Remark 10.3. (i) For $GL(N, \mathbb{R})/O(N)$, in [27] James obtains the expansion of $\psi_c(\nu, X)$ in a series of Jack polynomials, with a view towards statistical applications.

(ii) In [11, Section 8.9], the authors derive (in principle) a similar result to the one in Theorem 10.1 for the so-called Dunkl-kernel. Their method relies on Dunkl-Cherednik operators theory and Macdonald-type identity. To obtain our expansion, one needs to prove that the average over the Weyl group of the Dunkl-kernels can be expressed in terms of the Jack polynomials, with the appropriate explicit coefficients.

Connection to some of Hua's results. For $d = 2$, Theorem 10.1 can be written as

$$\lim_{\varsigma \to \infty} \frac{S_{\mathbf{m}\varsigma}(e^{\varsigma^{-1} x_1}, \dots, e^{\varsigma^{-1} x_N})}{S_{\mathbf{m}\varsigma}(1, \dots, 1)} = \sum_{\mu \in \mathscr{P}_N} \frac{S_\mu(x_1, \dots, x_N) S_\mu(m_1, \dots, m_N)}{\prod_{\square = (i,j) \in \mu} (N + j - i)}.$$

This limit contains all the results that can be found in [36, §5]. In fact, the above limit, i.e. for $d = 2$, is implicitly contained in Hua's book [17, §1.2]. By [17, Theorem 1.2.1], we know that if the power series

$$f_i(z) = \sum_{\kappa=0}^{\infty} a_\kappa^{(i)} z^\kappa$$

converges for $|z| < c$, then for $|z_1| < c, \cdots, |z_N| < c$ we have

$$\det(f_i(z_j))_{1 \le i,j \le N} = \sum_{\ell_1 > \ell_2 > \cdots \ell_N \ge 0} \det(a_{\ell_j}^{(i)}) \det(z_i^{\ell_j}). \tag{10.4}$$

Using (10.4) for the power series of the exponential function $e^{\varsigma^{-1} x_i (\varsigma \lambda_j + N - j)}$, and the fact that

$$S_\lambda(1, \dots, 1) = \frac{\mathrm{Vand}(\lambda_1 + N - 1, \dots, \lambda_{N-1} + 1, \lambda_N)}{\mathrm{Vand}(N - 1, \dots, 1, 0)},$$

one can deduce the limit formula. Here $\mathrm{Vand}(\cdots)$ denotes the Vandermonde determinant. Thus one can see that Theorem 10.1 is a generalization of Hua's limit formula for all $\wp > 0$.

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